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Co-isotropic and Legendre–Lagrangian submanifolds and conformal Jacobi morphisms

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Abstract. The notion of a co-isotropic and Legendre–Lagrangian submanifold of a Jacobi manifold is given. A characterization of conformal Jacobi morphisms and conformal Jacobi infinitesimal transformations is obtained as co-isotropic and Legendre–Lagrangian submanifolds of Jacobi manifolds.

1. Introduction

In [14], Tulczyjew characterized a locally Hamiltonian vector field on a symplectic manifold (M, Ω) as a Lagrangian submanifold of the symplectic manifold (TM, Ω^c), where TM is the tangent bundle of M and Ω^c is the complete or tangent lift of Ω to TM. This fact permitted the introduction of the notion of a generalized Hamiltonian system as a Lagrangian submanifold of (TM, Ω^c), and the discussion of, for instance, implicit differential equations (see, for instance, [11, 12]).

Recently, this result was extended by Grabowski and Urbánski [4] for Poisson manifolds. They proved that the tangent bundle TM of a Poisson manifold (M, Λ) is canonically endowed with a Poisson structure, namely, the complete lift Λ^c of Λ. Thus, they proved that a vector field X on M is a Poisson infinitesimal transformation (in other words, X is a derivation of the algebra $(C^\infty(M, \mathbb{R}), \{\,,\})$) if and only if its image $X(M)$ is a Lagrangian submanifold of (TM, Λ^c). Here, it is necessary to use a suitable definition of a Lagrangian submanifold. In fact, a submanifold S of a Poisson manifold is Lagrangian if and only if for every point x of S the intersection $T_xS \cap D_x$ is a Lagrangian subspace of D_x, where T_xS is the tangent space to S at x and D_x is the tangent space to the symplectic leaf by x. Concerning Poisson morphisms, it was proved by Weinstein in [16] (see also [15]) that a differentiable mapping $\phi : (M_1, \Lambda_1) \to (M_2, \Lambda_2)$ between the Poisson manifolds (M_1, Λ_1)

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and (M_2, Λ_2) is a Poisson morphism if and only if its graph is a co-isotropic submanifold of the Poisson manifold $(M_1 \times M_2, \Lambda_1 - \Lambda_2)$. This theorem extends the well known one for symplectic manifolds. These two results are also independently obtained by Sánchez de Alvárez [13].

The purpose of our paper is to extend the results for the case of Jacobi manifolds. Jacobi manifolds are more involved, and the extension is far from being trivial. In fact, if we start with a Jacobi manifold (M, Λ, E) and try to define in a natural way a similar structure on the tangent bundle TM, we would inevitably fail to do this. The reason is the intrinsic conformal character of Jacobi structures. In fact, the Hamiltonian vector fields in a Jacobi manifold are conformal Jacobi infinitesimal transformations. Thus, instead of using Jacobi morphisms we have to use conformal Jacobi morphisms. This implies that we are compelled to add an extra factor $\mathbb{R}$ to our Jacobi manifolds.

On the other hand, it is well known that the contact manifolds are canonical examples of Jacobi manifolds. In fact, the leaves of odd dimension of the characteristic foliation of a Jacobi manifold are contact manifolds (see [3] and section 2.2). Thus, as a first step, in section 3, we study the particular case of contact manifolds. In particular, we characterize the contact transformations (or equivalently, the conformal Jacobi isomorphisms) and the contact infinitesimal transformations (or equivalently, the conformal Jacobi infinitesimal transformations) in terms of Legendre submanifolds of contact manifolds (see theorems 3.7 and 3.13). The results obtained in this section provide a good motivation for the general study of conformal Jacobi morphisms between arbitrary Jacobi manifolds which will be introduced in sections 4–6. In these sections we generalize the results of section 3. More precisely, we prove the following results.

(1) Given two Jacobi manifolds (M_1, Λ_1, E_1) and (M_2, Λ_2, E_2), the product $M = M_1 \times M_2 \times \mathbb{R}$ is endowed with a Jacobi structure

$$\Lambda = \Lambda_1 + \frac{\partial}{\partial t} \wedge E_1 - \mathrm{e}^t \left(\Lambda_2 - \frac{\partial}{\partial t} \wedge E_2 \right) \qquad E = E_1.$$

Thus, given a mapping $\phi \colon M_1 \to M_2$ and a positive function $a \in C^\infty(M_1, \mathbb{R})$, we prove that the pair (a, ϕ) is a conformal Jacobi morphism if and only if $S = \{(x_1, \phi(x_1), \ln(1/a(x_1))) \in M_1 \times M_2 \times \mathbb{R} / x_1 \in M_1\}$ is a co-isotropic submanifold of (M, Λ, E) (theorem 5.3).

(2) In the same vein, if (M, Λ, E) is a Jacobi manifold, then $\mathbb{R} \times TM$ is endowed in a canonical way with a Jacobi structure given by

$$\bar{\Lambda} = \Lambda^{\mathrm{c}} + \frac{\partial}{\partial s} \wedge E^{\mathrm{c}} - s \left(\Lambda^{\mathrm{v}} + \frac{\partial}{\partial s} \wedge E^{\mathrm{v}} \right)$$

where Λ^{v} and E^{v} (respectively, Λ^{c} and E^{c}) are the vertical (respectively, complete) lifts of Λ and E to TM. Thus, we prove the following result. Given a vector field X and a function f on M, we denote by $f \times X : M \to \mathbb{R} \times TM$ the mapping $x \in M \to (f \times X)(x) = (f(x), X(x)) \in \mathbb{R} \times TM$. Then, the pair (f, X) is a conformal Jacobi infinitesimal transformation if and only if $(f \times X)(M)$ is a Legendre–Lagrangian submanifold of the Jacobi manifold $(\mathbb{R} \times TM, \bar{\Lambda}, E^{\mathrm{v}})$ (theorem 6.5).

Here, the concept of a co-isotropic submanifold is the natural extension to Jacobi manifolds of the notion of a co-isotropic submanifold in the setting of Poisson manifolds. Also, the Legendre–Lagrangian submanifold is the natural extension to Jacobi manifolds of the notions of Lagrangian and Legendre submanifolds in the setting of Poisson and contact manifolds.

The above results are proved by defining the so-called Poissonization of a Jacobi manifold (‘associated tangentially exact Poisson manifold’ in the terminology of Lichnerowicz [9]). The Poissonization of a contact manifold coincides with its symplectification. This technique permits us to obtain the above results as a consequence of the results of Grabowski and Urbánski, taking into account the relation between the co-isotropic and Legendre–Lagrangian submanifolds of a Jacobi manifold and the co-isotropic and Lagrangian submanifolds of its Poissonization (theorem 4.4).

All the manifolds considered throughout this paper are assumed to be connected.

2. Poisson morphisms and conformal Jacobi morphisms

2.1. Poisson morphisms

Let N be a C^∞ manifold. Denote by $x(N)$ the Lie algebra of the vector fields on N and by $C^\infty(N, \mathbb{R})$ the algebra of C^∞ real-valued functions on N. A Poisson bracket $\{\,,\,\}$ on N is a bilinear mapping $\{\,,\,\} : C^\infty(N, \mathbb{R}) \times C^\infty(N, \mathbb{R}) \to C^\infty(N, \mathbb{R})$ satisfying the following properties:

(1) (*Skew-symmetry*) $\{f, g\} = -\{g, f\}$.

(2) (*Leibniz rule*) $\{f, gh\} = \{f, g\}h + \{f, h\}g$.

(3) (*Jacobi's identity*) $\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0$.

The pair $(N, \{\,,\,\})$ will be called a Poisson manifold.

In [8], Lichnerowicz gave a more compact definition of a Poisson manifold. Define a 2-vector Λ on N by $\Lambda(\mathrm{d}f, \mathrm{d}g) = \{f, g\}$. Then $[\Lambda, \Lambda] = 0$, where $[,]$ is the Schouten–Nijenhuis bracket. Conversely, let Λ be a 2-vector on N and define a bracket of functions $\{f, g\} = \Lambda(\mathrm{d}f, \mathrm{d}g)$. Then, $\{,\}$ satisfies Jacobi's identity if and only if $[\Lambda, \Lambda] = 0$. Such a 2-vector Λ will be called a Poisson tensor.

The main examples of Poisson manifolds are symplectic manifolds. A symplectic manifold is a pair (N, Ω), where N is an even-dimensional manifold and Ω is a closed non-degenerate 2-form on N. We define a 2-vector Λ on N by

$$\Lambda(\alpha, \beta) = \Omega(\flat^{-1}(\alpha), \flat^{-1}(\beta)) \tag{1}$$

for $\alpha, \beta \in \Omega^1(N)$, where $\Omega^1(N)$ is the space of 1-forms on N and $\flat : x(N) \to \Omega^1(N)$ is the isomorphism of $C^\infty(N, \mathbb{R})$-modules defined by $\flat(X) = i_X\Omega$.

Let (N, Λ) be a Poisson manifold. Define a $C^\infty(N, \mathbb{R})$-linear mapping $\# : \Omega^1(N) \to x(N)$ as follows:

$$(\#\alpha)(\beta) = \Lambda(\alpha, \beta)$$

for $\alpha, \beta \in \Omega^1(N)$. If $f \in C^\infty(N, \mathbb{R})$, the vector field $X_f = \#(\mathrm{d}f)$ is called the Hamiltonian vector field associated with f.

Denote by D_x the subspace of T_xN generated by all the Hamiltonian vector fields evaluated at the point $x \in N$ or, in other words, $D_x = \#_x(T_x^*N)$. The distribution $x \in N \to D_x \subseteq T_xN$ is involutive (see [8]) and thus it defines a generalized foliation on N. Since the leaves of D are symplectic manifolds (see [8]), D is called the symplectic foliation of N.

Now, let $\phi : (N_1, \Lambda_1) \to (N_2, \Lambda_2)$ be a differentiable mapping between the Poisson manifolds (N_1, Λ_1) and (N_2, Λ_2). Suppose that $\{\,,\,\}_1$ (respectively, $\{\,,\,\}_2$) is the Poisson bracket on N_1 (respectively, N_2). Then, the mapping ϕ is said to be a Poisson morphism if $\{f_2, g_2\}_2 \circ \phi = \{f_2 \circ \phi, g_2 \circ \phi\}_1$ for $f_2, g_2 \in C^\infty(N_2, \mathbb{R})$ or, equivalently, if

$$\Lambda_1(\phi^*\alpha_2, \phi^*\beta_2) = \Lambda_2(\alpha_2, \beta_2) \circ \phi \tag{2}$$

for $\alpha_2, \beta_2 \in \Omega^1(N_2)$. If the Poisson morphism ϕ is a diffeomorphism then ϕ is called a Poisson isomorphism.

Remark 2.1. Let (N_1, Ω_1) and (N_2, Ω_2) be symplectic manifolds equipped with the Poisson structures associated with their symplectic structures. If a differentiable mapping $\phi : N_1 \to N_2$ is a Poisson morphism then it is necessarily a submersion. In the special case where N_1 and N_2 are of the same dimension, the mapping $\phi : N_1 \to N_2$ is a Poisson morphism if and only if it is a (local) symplectic isomorphism (that is, $\phi^*\Omega_2 = \Omega_1$). However, if $\dim N_1 > \dim N_2$ and ϕ is a Poisson morphism then ϕ is not a symplectic morphism (note that a symplectic morphism is necessarily an immersion) (for more details, see [7]).

A vector field X in a Poisson manifold (N, Λ) is said to be a Poisson infinitesimal transformation (see [4, 7, 8]) if its flow consists of Poisson isomorphisms, or, equivalently, if

$$\mathcal{L}_X \Lambda = 0 \tag{3}$$

where $\mathcal{L}$ is the Lie derivative on N. The Hamiltonian vector fields are Poisson infinitesimal transformations.

If (N, Ω) is a symplectic manifold and X a vector field on N, then X is a Poisson infinitesimal transformation if and only if X is a symplectic infinitesimal transformation (i.e. $\mathcal{L}_X \Omega = 0$).

A submanifold S of a Poisson manifold (N, Λ) is called co-isotropic [15, 16] if

$$\#_x (T_x S)^0 \subseteq T_x S \tag{4}$$

for $x \in S$, $(T_x S)^0$ being the annihilator subspace of $T_x S$. The submanifold S is said to be Lagrangian [4, 15] if

$$\#_x (T_x S)^0 = T_x S \cap D_x. \tag{5}$$

The above definitions generalize the usual definitions of co-isotropic and Lagrangian submanifold in a symplectic manifold.

We also have:

Theorem 2.2. [16]. Let $\phi : (N_1, \Lambda_1) \to (N_2, \Lambda_2)$ be a differentiable mapping between the Poisson manifolds (N_1, Λ_1) and (N_2, Λ_2). Then, ϕ is a Poisson morphism if and only if $\mathrm{Graph}\,\phi$ is a co-isotropic submanifold of the Poisson manifold $(N_1 \times N_2, \Lambda)$, Λ being the 2-vector on $N_1 \times N_2$ given by $\Lambda = \Lambda_1 - \Lambda_2$.

Theorem 2.3. [4, 13]. Let X be a vector field on a Poisson manifold (N, Λ). Then:

(i) the complete lift Λ^c to TN of Λ is a Poisson structure on TN;

(ii) X is a Poisson infinitesimal transformation if and only if $X(N)$ is a Lagrangian submanifold of (TN, Λ^c).

Remark 2.4. If (N, Ω) is a symplectic manifold then the complete lift Ω^c of Ω to TN is a symplectic 2-form on TN. Moreover, if Λ is the Poisson structure on N associated with the symplectic 2-form Ω then the Poisson structure on TN associated with the symplectic 2-form Ω^c is just Λ^c. Thus, theorems 2.2 and 2.3 generalize for the Poisson manifolds the corresponding results on symplectic isomorphisms and symplectic infinitesimal transformations (see [14]).

2.2. Conformal Jacobi morphisms

A Jacobi structure on M is a pair (Λ, E) where Λ is a 2-vector and E a vector field on M satisfying

$$[\Lambda, \Lambda] = 2E \wedge \Lambda \qquad \mathcal{L}_E \Lambda = [E, \Lambda] = 0. \tag{6}$$

The manifold M endowed with a Jacobi structure is called a Jacobi manifold. If (M, Λ, E) is a Jacobi manifold we can define a bracket of functions (called a Jacobi bracket) as follows:

$$\{f, g\} = \Lambda(\mathrm{d}f, \mathrm{d}g) + fE(g) - gE(f) \qquad \text{for all } f, g \in C^\infty(M, \mathbb{R}). \tag{7}$$

The mapping $\{\,,\}\colon C^\infty(M, \mathbb{R}) \times C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ is bilinear, skew-symmetric, satisfies the Jacobi's identity and

$$\text{support } \{f, g\} \subset \text{support } f \cap \text{support } g.$$

Thus, the space $C^\infty(M, \mathbb{R})$ endowed with the Jacobi bracket is a local Lie algebra in the sense of Kirillov (see [6]). Conversely, a structure of local Lie algebra on the space $C^\infty(M, \mathbb{R})$ of real-valued functions on a manifold M determines a Jacobi structure on M (see [5, 6]).

If the vector field E vanishes, then $\{\,,\}$ satisfies the Leibniz rule and it is a Poisson bracket on M. In this case, (M, Λ) is a Poisson manifold. The Jacobi manifolds were introduced by Lichnerowicz [9].

The canonical examples of Jacobi manifolds (apart from symplectic and Poisson manifolds) are the contact and locally conformal symplectic manifolds.

Let M be a $(2m + 1)$-dimensional manifold and η a 1-form on M. We say that η is a contact 1-form if $\eta \wedge (\mathrm{d}\eta)^m \neq 0$ at every point. In such a case, (M, η) is termed a contact manifold [1, 2]. A contact manifold (M, η) is a Jacobi manifold. In fact, the pair (Λ, E) is a Jacobi structure on M, where

$$\Lambda(\alpha, \beta) = \mathrm{d}\eta(\flat^{-1}(\alpha), \flat^{-1}(\beta)) \qquad E = \flat^{-1}(\eta) \tag{8}$$

for $\alpha, \beta \in \Omega^1(M)$, with $\flat : x(M) \to \Omega^1(M)$ the isomorphism of $C^\infty(M, \mathbb{R})$-modules defined by $\flat(X) = i_X \mathrm{d}\eta + \eta(X)\eta$. The vector field E is called the Reeb vector field of M and it is characterized by the relations $i_E\eta = 1$ and $i_E\, \mathrm{d}\eta = 0$.

On the other hand, let us recall that an almost symplectic manifold is a pair (M, Ω), where M is an even-dimensional manifold and Ω is a non-degenerate 2-form on M. An almost symplectic manifold is said to be locally conformal symplectic (LCS) if there exists a closed 1-form ω such that $\mathrm{d}\Omega = \omega \wedge \Omega$. The 1-form ω is called the Lee 1-form of M. If (M, Ω) is a LCS manifold then the pair (Λ, E) is a Jacobi structure on M, where

$$\Lambda(\alpha, \beta) = \Omega(\flat^{-1}(\alpha), \flat^{-1}(\beta)) \qquad E = \flat^{-1}\omega$$

for $\alpha, \beta \in \Omega^1(M)$, with $\flat : x(M) \to \Omega^1(M)$ the isomorphism of $C^\infty(M, \mathbb{R})$-modules defined by $\flat(X) = i_X\Omega$.

Now, let (M, Λ, E) be a Jacobi manifold. Define a $C^\infty(M, \mathbb{R})$-linear mapping $\# : \Omega^1(M) \to x(M)$ by

$$(\#\alpha)(\beta) = \Lambda(\alpha, \beta)$$

for $\alpha, \beta \in \Omega^1(M)$. Then, if $f \in C^\infty(M, \mathbb{R})$, the vector field X_f given by $X_f = \#(\mathrm{d}f) + fE$, is called the Hamiltonian vector field associated with f. It should be noted that the Hamiltonian vector field associated with the constant function 1 is just E. A direct computation shows that $[X_f, X_g] = X_{\{f,g\}}$ [9, 10]. Denote by D_x the subspace of T_xM generated by all the Hamiltonian vector fields evaluated at the point $x \in M$. In other words, $D_x = \#_x(T_x^*M) + \langle E_x \rangle$. Since D is involutive, one easily follows that D defines a generalized foliation on M, which is called the characteristic foliation. It is proved that the leaves of D are contact or LCS manifolds (for a detailed study we refer to [3]).

Next, we recall the definition of conformally equivalent Jacobi structures (see [7, 9]).

Let (M, Λ, E) be a Jacobi manifold and a a function without zeros that belongs to $C^\infty(M, \mathbb{R})$. Let us consider the 2-vector Λ_a and the vector field E_a on M given by

$$\Lambda_a = a\Lambda \qquad E_a = \#(\mathrm{d}a) + aE = X_a.$$

Then, the pair (Λ_a, E_a) is a Jacobi structure on M. The brackets $\{\,,\}$ and $\{\,,\}_a$ are related by

$$\{f, g\}_a = \frac{1}{a}\{af, ag\} \qquad \forall f, g \in C^\infty(M, \mathbb{R}).$$

We say that the Jacobi structures (Λ, E) and (Λ_a, E_a) are conformally equivalent.

Remark 2.5. Since all manifolds are assumed to be connected we have that a is either a positive or negative function. For the sake of symplicity, and without loss of generality, we will always suppose that a is a positive function.

Let $\phi\colon (M_1, \Lambda_1, E_1) \to (M_2, \Lambda_2, E_2)$ be a differentiable mapping between the Jacobi manifolds (M_1, Λ_1, E_1) and (M_2, Λ_2, E_2). Suppose that $\{\,,\}_1$ (respectively, $\{\,,\}_2$) is the Jacobi bracket on M_1 (respectively, M_2).

The mapping ϕ is said to be a Jacobi morphism [3] if $\{f_2, g_2\}_2 \circ \phi = \{f_2 \circ \phi, g_2 \circ \phi\}_1$ for $f_2, g_2 \in C^\infty(M_2, \mathbb{R})$ or, equivalently, if

$$\Lambda_1(\phi^*\alpha_2, \phi^*\beta_2) = \Lambda_2(\alpha_2, \beta_2) \circ \phi \qquad \phi_* E_1 = E_2 \tag{9}$$

for $\alpha_2, \beta_2 \in \Omega^1(M_2)$.

Now, if a is a positive function on M_1 then the pair (a, ϕ) is called a conformal Jacobi morphism [3] if the mapping ϕ is a Jacobi morphism between the Jacobi manifolds $(M_1, (\Lambda_1)_a, (E_1)_a)$ and (M_2, Λ_2, E_2). The conformal Jacobi isomorphisms are the conformal Jacobi morphisms (a, ϕ) such that ϕ is a diffeomorphism.

A vector field X on a Jacobi manifold (M, Λ, E) is said to be a Jacobi infinitesimal transformation if

$$\mathcal{L}_X\Lambda = 0 \qquad \mathcal{L}_X E = 0 \tag{10}$$

and it is a conformal Jacobi infinitesimal transformation if there exists $g \in C^\infty(M, \mathbb{R})$ such that [3, 7, 9]

$$\mathcal{L}_X\Lambda = g\Lambda \qquad \mathcal{L}_X E = \#(\mathrm{d}g) + gE = X_g. \tag{11}$$

In what follows, the pair (g, X) will be called a conformal Jacobi infinitesimal transformation.

Note that the Hamiltonian vector fields are conformal Jacobi infinitesimal transformations. In fact, if $f \in C^\infty(M, \mathbb{R})$ then

$$\mathcal{L}_{X_f}\Lambda = -E(f)\Lambda \qquad \mathcal{L}_{X_f}E = -\#(\mathrm{d}(E(f))) - E(f)E = -X_{E(f)}. \tag{12}$$

In particular, if f is a basic function (that is, $E(f) = 0$) then X_f is a Jacobi infinitesimal transformation.

3. Contact transformations and Legendre submanifolds

In this section, we will obtain some results on a particular class of conformal Jacobi morphisms between contact manifolds, the contact transformations. Also, we will study the relation between the contact transformations and the Legendre submanifolds. The general study on conformal Jacobi morphisms between arbitrary Jacobi manifolds will be discussed in sections 4–6. However, the results obtained in this section provide a good motivation for such a study.

3.1. Legendre submanifolds in a contact manifold

Let (M, η) be a $(2m + 1)$-dimensional contact manifold. Denote by (Λ, E) its associated Jacobi structure, and by $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ the isomorphism of $C^\infty(M, \mathbb{R})$-modules defined by $\flat(X) = i_X \mathrm{d}\eta + \eta(X)\eta$, for $X \in \mathfrak{X}(M)$.

If $x \in M$ then a direct computation, using (8), proves that

$$\#_x(\alpha) = -\flat_x^{-1}(\alpha) + \alpha(E_x)E_x \qquad (13)$$

for $\alpha \in T_x^*M$. Thus, the linear mapping $\#_x : T_x^*M \to \langle\eta_x\rangle^0$ is an epimorphism and $\operatorname{Ker} \#_x = \langle\eta_x\rangle$. In particular, the linear mapping $\#_x : \langle E_x\rangle^0 \to \langle\eta_x\rangle^0$ is an isomorphism.

Now, let F be the $2m$-dimensional distribution on M given by $\eta = 0$. F is called the contact distribution of M. It is well known (see, for example, [2]) that there exist integral submanifolds of the contact distribution F of dimension m but of no higher dimension.

Definition 3.1. A submanifold S of M is said to be a Legendre submanifold [7] if it is a m-dimensional integral submanifold of the contact distribution.

Note that if S is a Legendre submanifold of M and $x \in S$ then

$$(\flat_x(u))(v) = \mathrm{d}\eta_x(u, v) = 0$$

for $u, v \in T_xS$. Using this fact and (13), we conclude that:

Proposition 3.2. Let (M, η) be a contact manifold and S a submanifold of M. Then, S is a Legendre submanifold of M if and only if

$$\#_x(T_xS)^0 = T_xS$$

for $x \in S$.

3.2. Contact transformations and Legendre submanifolds

We recall the definition of a contact transformation (see [2, 7, 9]).

Definition 3.3. Let $\phi : (M_1, \eta_1) \to (M_2, \eta_2)$ be a diffeomorphism between the contact manifolds (M_1, η_1) and (M_2, η_2) and suppose that a is a positive function on M_1. The pair (a, ϕ) is said to be a contact transformation if

$$\phi^*\eta_2 = a\eta_1.$$

The conformal Jacobi isomorphisms between two contact manifolds are just the contact transformations. In fact, we have (see [9]):

Proposition 3.4. Let $\phi : (M_1, \eta_1) \to (M_2, \eta_2)$ be a differentiable mapping between the contact manifolds (M_1, η_1) and (M_2, η_2) and suppose that a is a positive function on M_1. Then, the pair (a, ϕ) is a conformal Jacobi isomorphism if and only if the pair $(1/a, \phi)$ is a contact transformation.

Next, we will show that the product of two contact manifolds with $\mathbb{R}$ is a contact manifold. This result will be useful in the following.

Proposition 3.5. Let (M_1, η_1) and (M_2, η_2) be contact manifolds with Reeb vector fields E_1 and E_2, respectively. If t is the usual coordinate on $\mathbb{R}$ then the 1-form η on the product manifold $M_1 \times M_2 \times \mathbb{R}$ given by

$$\eta = \eta_1 - \mathrm{e}^{-t}\eta_2 \tag{14}$$

is a contact 1-form. The Reeb vector field of $(M_1 \times M_2 \times \mathbb{R}, \eta)$ is E_1.

Proof. It follows from a direct computation. □

Remark 3.6. Using (8) and proposition 3.5, we obtain that the Jacobi structure (Λ, E) on the contact manifold $(M_1 \times M_2 \times \mathbb{R}, \eta)$ is

$$\Lambda = \Lambda_1 + \frac{\partial}{\partial t} \wedge E_1 - \mathrm{e}^t \left(\Lambda_2 - \frac{\partial}{\partial t} \wedge E_2 \right) \qquad E = E_1$$

where (Λ_1, E_1) and (Λ_2, E_2) are the Jacobi structures of M_1 and M_2, respectively.

The following result justifies the definition of the contact 1-form η on the product manifold $M_1 \times M_2 \times \mathbb{R}$.

Theorem 3.7. Let $\phi : (M_1, \eta_1) \to (M_2, \eta_2)$ be a diffeomorphism between the contact manifolds (M_1, η_1) and (M_2, η_2). Suppose that a is a positive function on M_1, and S is the submanifold of the product manifold $M_1 \times M_2 \times \mathbb{R}$ defined by

$$S = \left\{ \left(x_1, \phi(x_1), \ln \frac{1}{a(x_1)} \right) \in M_1 \times M_2 \times \mathbb{R} / x_1 \in M_1 \right\}.$$

Then, the pair (a, ϕ) is a conformal Jacobi isomorphism if and only if S is a Legendre submanifold of $(M_1 \times M_2 \times \mathbb{R}, \eta)$, η being the contact 1-form on $M_1 \times M_2 \times \mathbb{R}$ given by (14).

Proof. If x_1 is a point of M_1 and $z = (x_1, \phi(x_1), \ln(1/a(x_1)))$ is the corresponding point of the submanifold S then the tangent space T_zS is the subspace of $T_z(M_1 \times M_2 \times \mathbb{R}) \approx T_{x_1}M_1 \oplus T_{\phi(x_1)}M_2 \oplus T_{\ln(1/a(x_1))}\mathbb{R}$ given by

$$T_zS = \left\{ \left(v_1, \phi_*^{x_1} v_1, \frac{-v_1(a)}{a(x_1)} \left(\frac{\partial}{\partial t} \right) \Big|_{\ln(1/a(x_1))} \right) \in T_z(M_1 \times M_2 \times \mathbb{R}) / v_1 \in T_{x_1}M_1 \right\}.$$

Thus, since $\dim S = \dim M_1 = \dim M_2$, we prove the result using (14), definition 3.3 and proposition 3.4. □

From theorem 3.7, we conclude:

Corollary 3.8. Let $\phi : (M_1, \eta_1) \to (M_2, \eta_2)$ be a diffeomorphism between the contact manifolds (M_1, η_1) and (M_2, η_2). Then, ϕ is a Jacobi isomorphism if and only if Graph $\phi \times \{0\}$ is a Legendre submanifold of $(M_1 \times M_2 \times \mathbb{R}, \eta)$, η being the contact 1-form on $M_1 \times M_2 \times \mathbb{R}$ given by (14).

3.3. Contact infinitesimal transformations and Legendre submanifolds

We recall the definition of a contact infinitesimal transformation (see [7, 9]).

Definition 3.9. Let (M, η) be a contact manifold and X a vector field on M. If f is a C^∞ function on M then the pair (f, X) is said to be a contact infinitesimal transformation if

$$\mathcal{L}_X \eta = f\eta.$$

The conformal Jacobi infinitesimal transformations in a contact manifold are just the contact infinitesimal transformations. In fact, we have [9] the following.

Proposition 3.10. Let (M, η) be a contact manifold, X a vector field on M and f a C^∞ function on M. Then, the pair (f, X) is a conformal Jacobi infinitesimal transformation if and only if the pair $(-f, X)$ is a contact infinitesimal transformation.

Now, we will prove that the product of $\mathbb{R}$ with the tangent bundle TM of a contact manifold M is also a contact manifold.

Proposition 3.11. Let (M, η) be a contact manifold with Reeb vector field E and $\bar{\eta}$ the 1-form on the product manifold $\mathbb{R} \times TM$ defined by

$$\bar{\eta} = \eta^{\mathrm{c}} + s\eta^{\mathrm{v}} \tag{15}$$

where s is the usual coordinate on $\mathbb{R}$ and η^{c} (respectively, η^{v}) is the complete (respectively, vertical) lift of η to TM. Then, $(\mathbb{R} \times TM, \bar{\eta})$ is a contact manifold with Reeb vector field the vertical lift E^{v} of E to TM.

Proof. If X is a vector field on M we will denote by X^{c} (respectively, X^{v}) the complete (respectively, vertical) lift of X to TM.

Let $\bar{\flat} : \mathfrak{X}(\mathbb{R}\times TM) \to \Omega^1(\mathbb{R}\times TM)$ be the homomorphism of $C^\infty(\mathbb{R}\times TM, \mathbb{R})$-modules defined by

$$\bar{\flat}(\bar{X}) = i_{\bar{X}}\, \mathrm{d}\bar{\eta} + \bar{\eta}(\bar{X})\bar{\eta}$$

for $\bar{X} \in \mathfrak{X}(\mathbb{R} \times TM)$. If X is a vector field on M such that $\eta(X) = 0$ then, using (15) and the results of [17], we have that

$$\bar{\flat}(X^{\mathrm{v}}) = \flat(X)^{\mathrm{v}} \qquad \bar{\flat}(X^{\mathrm{c}}) = \flat(X)^{\mathrm{c}} + s\flat(X)^{\mathrm{v}}$$

$$\bar{\flat}(E^{\mathrm{v}}) = \eta^{\mathrm{c}} + s\eta^{\mathrm{v}} = \bar{\eta} \qquad \bar{\flat}(E^{\mathrm{c}}) = -\mathrm{d}s + s\bar{\eta} \qquad \bar{\flat}\left(\frac{\partial}{\partial s}\right) = \eta^{\mathrm{v}}.$$

Thus $\bar{\flat}$ is an isomorphism of $C^\infty(\mathbb{R} \times TM, \mathbb{R})$-modules which implies that $(\mathbb{R} \times TM, \bar{\eta})$ is a contact manifold (see [1]). Moreover, since $\bar{\flat}(E^{\mathrm{v}}) = \bar{\eta}$, we deduce that E^{v} is the Reeb vector field of the contact manifold $(\mathbb{R} \times TM, \bar{\eta})$. □

Remark 3.12. Using (8), proposition 3.11 and the results of [17] we obtain that the Jacobi structure $(\bar{\Lambda}, \bar{E})$ on the contact manifold $(\mathbb{R} \times TM, \bar{\eta})$ is

$$\bar{\Lambda} = \Lambda^{\mathrm{c}} + \frac{\partial}{\partial s} \wedge E^{\mathrm{c}} - s\left(\Lambda^{\mathrm{v}} + \frac{\partial}{\partial s} \wedge E^{\mathrm{v}}\right) \qquad \bar{E} = E^{\mathrm{v}}.$$

The following result justifies the definition of the contact 1-form $\bar{\eta}$ on the product manifold $\mathbb{R} \times TM$.

Theorem 3.13. Let (M, η) be a contact manifold. Suppose that X is a vector field on M and denote by $f \times X : M \to \mathbb{R} \times TM$ the mapping

$$x \in M \to (f \times X)(x) = (f(x), X(x)) \in \mathbb{R} \times TM$$

f being a C^∞ function on M. Then, the pair (f, X) is a conformal Jacobi infinitesimal transformation if and only if $(f \times X)(M)$ is a Legendre submanifold of the contact manifold $(\mathbb{R} \times TM, \bar{\eta})$, where $\bar{\eta}$ is the 1-form on $\mathbb{R} \times TM$ given by (15).

Proof. If $S = (f \times X)(M)$, x is a point of M and $z = (f(x), X_x) \in S$ then the tangent space $T_z S$ is the subspace of $T_z(\mathbb{R} \times TM) \approx T_{f(x)}\mathbb{R} \oplus T_{X_x}(TM)$ given by

$$T_z S = \left\{ \left(v(f) \frac{\partial}{\partial t}\bigg|_{f(x)}, X_*^x(v) \right) \in T_z(\mathbb{R} \times TM)/v \in T_x M \right\}. \tag{16}$$

On the other hand, from the results of [17], we deduce that

$$X^*(\eta^{\mathrm{c}}) = \mathcal{L}_X \eta \qquad \eta^{\mathrm{v}}_{X_x} \circ X_*^x = \eta_x. \tag{17}$$

Therefore, since $\dim S = \dim M$, we prove the result using (15)–(17), definition 3.9 and proposition 3.10. □

Finally, from theorem 3.13, we conclude:

Corollary 3.14. Let (M, η) be a contact manifold and X a vector field on M. Then, X is a Jacobi infinitesimal transformation if and only if $\{0\} \times X(M)$ is a Legendre submanifold of $(\mathbb{R} \times TM, \bar{\eta})$, $\bar{\eta}$ being the contact 1-form on $\mathbb{R} \times TM$ given by (15).

3.4. The symplectification of a contact manifold

Let M be a $(2m + 1)$-dimensional manifold and η a 1-form on M. We consider on the product manifold $M \times \mathbb{R}$ the 2-form Ω given by

$$\Omega = \mathrm{e}^t \,\mathrm{d}\eta + \mathrm{e}^t \,\mathrm{d}t \wedge \eta. \tag{18}$$

From (18), we deduce that η is a contact 1-form on M if and only if Ω is a symplectic 2-form on $M \times \mathbb{R}$.

If (M, η) is a contact manifold then the symplectic manifold $(M \times \mathbb{R}, \Omega)$ is called the symplectification of M (see [7]).

Suppose that (M, η) is a contact manifold and denote by (Λ, E) its associated Jacobi structure, and by $\tilde{\Lambda}$ the Poisson structure on the symplectification $(M \times \mathbb{R}, \Omega)$. A direct computation, using (1) and (18), proves that

$$\tilde{\Lambda} = \mathrm{e}^{-t} \Lambda + \mathrm{e}^{-t} \frac{\partial}{\partial t} \wedge E. \tag{19}$$

Next, we will study the relation between the Legendre submanifolds of M and the Lagrangian submanifolds of $M \times \mathbb{R}$.

Theorem 3.15. Let (M, η) be a contact manifold, S a submanifold of M and $(M \times \mathbb{R}, \Omega)$ the symplectification of M. Then, S is a Legendre submanifold of M if and only if $S \times \mathbb{R}$ is a Lagrangian submanifold of the symplectic manifold $(M \times \mathbb{R}, \Omega)$.

Proof. If N is a $2n$-dimensional symplectic manifold and D is the symplectic foliation of N, then $D_x = T_x N$ for $x \in N$. Thus, a submanifold $\tilde{S}$ of N is Lagrangian if and only if $\dim \tilde{S} = n$ and $\#_x(T_x\tilde{S})^0 \subseteq T_x\tilde{S}$ for $x \in \tilde{S}$ (see (5)).

On the other hand, if $x_0 \in M$ and $t_0 \in \mathbb{R}$ then, under the identification $T^*_{(x_0,t_0)}(M \times \mathbb{R}) \approx T^*_{x_0}M \oplus T^*_{t_0}\mathbb{R}$, we have that $(T_{x_0}S)^0 \oplus \{0\} = T_{(x_0,t_0)}(S \times \mathbb{R})^0$.

Using these facts, (13), (19) and proposition 3.2, we prove the result. □

Now, we will show that a conformal Jacobi isomorphism between two contact manifolds induces a symplectic isomorphism (or equivalently a Poisson isomorphism) between the corresponding symplectifications. In fact, we obtain:

Theorem 3.16. Let $\phi : (M_1, \eta_1) \to (M_2, \eta_2)$ be a diffeomorphism between the contact manifolds (M_1, η_1) and (M_2, η_2). Suppose that a is a positive function on M_1 and denote by $\tilde{\phi}_a : M_1 \times \mathbb{R} \to M_2 \times \mathbb{R}$ the diffeomorphism defined by

$$\tilde{\phi}_a(x_1, p) = (\phi(x_1), p + \ln(a(x_1))). \tag{20}$$

Then, the pair (a, ϕ) is a conformal Jacobi isomorphism if and only if $\tilde{\phi}_a$ is a symplectic isomorphism between the symplectifications $(M_1 \times \mathbb{R}, \Omega_1)$ and $(M_2 \times \mathbb{R}, \Omega_2)$ of M_1 and M_2, respectively.

Proof. It follows from definition 3.3, proposition 3.4, (18) and (20). □

Finally, we will see that a conformal Jacobi infinitesimal transformation in a contact manifold induces a symplectic infinitesimal transformation of the corresponding symplectification.

Theorem 3.17. Let (M, η) be a contact manifold, X a vector field and f a C^∞ function on M. Then, the pair (f, X) is a conformal Jacobi infinitesimal transformation if and only if the vector field $\tilde{X}_f$ on $M \times \mathbb{R}$ given by

$$\tilde{X}_f = X + f\frac{\partial}{\partial t} \tag{21}$$

is a symplectic infinitesimal transformation of the symplectification $(M \times \mathbb{R}, \Omega)$ of M.

Proof. It follows using definition 3.9, proposition 3.10, (18) and (21). □

4. Co-isotropic and Legendre–Lagrangian submanifolds in a Jacobi manifold

In this section, we introduce and characterize the notion of a co-isotropic and Legendre–Lagrangian submanifold in a Jacobi manifold.

Definition 4.1. Let (M, Λ, E) be a Jacobi manifold with characteristic foliation D and S a submanifold of M. The submanifold S is said to be:

(i) co-isotropic if

$$\#_x(T_xS)^0 \subseteq T_xS \tag{22}$$

for $x \in S$;

(ii) Legendre–Lagrangian if

$$\#_x(T_xS)^0 = T_xS \cap D_x \tag{23}$$

for $x \in S$.

Remark 4.2. (i) Definition 4.1 generalizes for Jacobi manifolds the notion of a co-isotropic and Lagrangian submanifold in a Poisson manifold (see (4), (5), (22) and (23)).

(ii) Let (M, η) be a contact manifold and S a submanifold of M. Suppose that (Λ, E) is the associated Jacobi structure on M. Then, S is a Legendre submanifold of the contact manifold (M, η) if and only if S is a Legendre–Lagrangian submanifold of the Jacobi manifold (M, Λ, E) (see (23) and proposition 3.2). Note that, in this case, $D_x = T_x M$ for $x \in M$.

Next, we prove a result which will be useful in the following.

Lemma 4.3. Let (M, Λ, E) be a Jacobi manifold and S a Legendre–Lagrangian submanifold of M. Suppose that x_0 is a point of S such that $E_{x_0} \notin \#_{x_0}(T^*_{x_0}M)$. Then, there exists a 1-form η_{x_0} on M at x_0 such that $\eta_{x_0} \in (T_{x_0}S)^0$, $\eta_{x_0}(E_{x_0}) = 1$ and $\#_{x_0}(\eta_{x_0}) = 0$.

Proof. Using (23), we deduce that $T_{x_0}S \cap D_{x_0} = T_{x_0}S \cap \#_{x_0}(T^*_{x_0}M)$. Assume that the rank of Λ at x_0 is $2r$. Then,

$$\Lambda_{x_0} = \sum_{i=1}^{r} u_i \wedge v_i \tag{24}$$

with $\{u_1, \ldots, u_r, v_1, \ldots, v_r\}$ a linearly independent system in $T_{x_0}M$. From (24), we obtain that

$$\#_{x_0}(T^*_{x_0}M) = \langle u_1, \ldots, u_r, v_1, \ldots, v_r \rangle. \tag{25}$$

Let W_{x_0} be a subspace of $T_{x_0}S$ such that

$$T_{x_0}S = (T_{x_0}S \cap D_{x_0}) \oplus W_{x_0} = (T_{x_0}S \cap \#_{x_0}(T^*_{x_0}M)) \oplus W_{x_0}. \tag{26}$$

If $\dim W_{x_0} = s$ and $\{w_1, \ldots, w_s\}$ is a basis of W_{x_0} then, using (25), (26) and the fact that $E_{x_0} \notin \#_{x_0}(T^*_{x_0}M)$, we have that $\{u_1, \ldots, u_r, v_1, \ldots, v_r, E_{x_0}, w_1, \ldots, w_s\}$ is also a linearly independent system in $T_{x_0}M$.

Now, suppose that $\{u_1, \ldots, u_r, v_1, \ldots, v_r, E_{x_0}, w_1, \ldots, w_s, z_1, \ldots, z_p\}$ is a basis of $T_{x_0}M$ and that $\{\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_r, \eta_{x_0}, \gamma_1, \ldots, \gamma_s, \nu_1, \ldots, \nu_p\}$ is the dual basis of $T^*_{x_0}M$. Then, from (24)–(26), we conclude that $\eta_{x_0} \in (T_{x_0}S)^0$, $\eta_{x_0}(E_{x_0}) = 1$ and $\#_{x_0}(\eta_{x_0}) = 0$. □

Let M be a differentiable manifold, Λ a 2-vector and E a vector field on M. We consider on the product manifold $M \times \mathbb{R}$ the 2-vector $\tilde{\Lambda}$ given by

$$\tilde{\Lambda} = \mathrm{e}^{-t}\Lambda + \mathrm{e}^{-t}\frac{\partial}{\partial t} \wedge E \tag{27}$$

t being the usual coordinate on $\mathbb{R}$.

We have that the pair (Λ, E) is a Jacobi structure on M if and only if $\tilde{\Lambda}$ is a Poisson structure on $M \times \mathbb{R}$ (see [9]).

If (M, Λ, E) is a Jacobi manifold then the product manifold $M \times \mathbb{R}$ with the structure $\tilde{\Lambda}$ was called by Lichnerowicz [9] the tangentially exact Poisson manifold associated with M. In what follows, and taking into account the fact that for a contact manifold (M, η), $\tilde{\Lambda}$ is just its symplectification, we will call it the Poissonization of the Jacobi manifold (M, Λ, E).

Theorem 4.4. Let (M, Λ, E) be a Jacobi manifold, S a submanifold of M and $(M \times \mathbb{R}, \tilde{\Lambda})$ the Poissonization of M. Then:

(i) S is a co-isotropic submanifold of M if and only if $S \times \mathbb{R}$ is a co-isotropic submanifold of the Poisson manifold $(M \times \mathbb{R}, \tilde{\Lambda})$;

(ii) S is a Legendre–Lagrangian submanifold of M if and only if $S \times \mathbb{R}$ is a Lagrangian submanifold of the Poisson manifold $(M \times \mathbb{R}, \tilde{\Lambda})$.

Proof. If $x_0 \in S$ and $t_0 \in \mathbb{R}$ then, under the identification $T^*_{(x_0,t_0)}(M \times \mathbb{R}) \approx T^*_{x_0}M \oplus T^*_{t_0}\mathbb{R}$, we have that $(T_{x_0}S)^0 \oplus \{0\} = T_{(x_0,t_0)}(S \times \mathbb{R})^0$. Using this fact, (4), (22) and (27) we prove (i).

Now, suppose that $S \times \mathbb{R}$ is a Lagrangian submanifold of the Poisson manifold $(M \times \mathbb{R}, \tilde{\Lambda})$. Then, from (5), (23) and (27), we deduce that S is a Legendre–Lagrangian submanifold of M.

Conversely, let S be a Legendre–Lagrangian submanifold of M. If $x_0 \in S$ and $t_0 \in \mathbb{R}$ then, using (23) and (27), we obtain that $\tilde{\#}_{(x_0,t_0)}(\alpha) \in T_{(x_0,t_0)}(S \times \mathbb{R}) \approx T_{x_0}S \oplus T_{t_0}\mathbb{R}$, for $\alpha \in T_{(x_0,t_0)}(S \times \mathbb{R})^0 \approx (T_{x_0}S)^0 \oplus \{0\}$. This shows that

$$\tilde{\#}_{(x_0,t_0)}(T_{(x_0,t_0)}(S \times \mathbb{R})^0) \subseteq \tilde{\#}_{(x_0,t_0)}(T^*_{(x_0,t_0)}(M \times \mathbb{R})) \cap T_{(x_0,t_0)}(S \times \mathbb{R}).$$

Next, we will prove that

$$\tilde{\#}_{(x_0,t_0)}(T^*_{(x_0,t_0)}(M \times \mathbb{R})) \cap T_{(x_0,t_0)}(S \times \mathbb{R}) \subseteq \tilde{\#}_{(x_0,t_0)}(T_{(x_0,t_0)}(S \times \mathbb{R})^0)$$

which implies that $S \times \mathbb{R}$ is a Lagrangian submanifold of the Poisson manifold $(M \times \mathbb{R}, \tilde{\Lambda})$.

Let α be a 1-form on M at x_0 and λ a real number such that $\tilde{\#}_{(x_0,t_0)}(\alpha + \lambda\, \mathrm{d}t|_{t_0}) \in T_{(x_0,t_0)}(S \times \mathbb{R}) \approx T_{x_0}S \oplus T_{t_0}\mathbb{R}$. From (23) and (27), we deduce that there exists $\beta \in (T_{x_0}S)^0$ such that

$$\#_{x_0}(\beta) = \#_{x_0}(\alpha) + \lambda E_{x_0}. \tag{28}$$

We will see that there exists $\gamma \in (T_{x_0}S)^0$ satisfying

$$\#_{x_0}(\gamma) = \#_{x_0}(\alpha) + \lambda E_{x_0} \qquad \gamma(E_{x_0}) = \alpha(E_{x_0}) \tag{29}$$

which shows that $\tilde{\#}_{(x_0,t_0)}(\alpha + \lambda\, \mathrm{d}t|_{t_0}) \in \tilde{\#}_{(x_0,t_0)}(T_{(x_0,t_0)}(S \times \mathbb{R})^0)$ (see (27)).

We distinguish two cases.

(a) Suppose that $E_{x_0} \in \#_{x_0}(T^*_{x_0}M)$. In such a case, there is a 1-form ω_{x_0} on M at x_0 such that $\#_{x_0}(\omega_{x_0}) = E_{x_0}$. We have that $\omega_{x_0}(E_{x_0}) = 0$ and, using (28), we conclude that $\alpha(E_{x_0}) = \beta(E_{x_0})$. Thus, the 1-form $\beta \in (T_{x_0}S)^0$ satisfies (29).

(b) Suppose that $E_{x_0} \notin \#_{x_0}(T^*_{x_0}M)$. In this case, we consider a 1-form $\eta_{x_0} \in (T_{x_0}S)^0$ such that $\eta_{x_0}(E_{x_0}) = 1$ and $\#_{x_0}(\eta_{x_0}) = 0$ (see lemma 4.3). Then, the 1-form $\gamma = \beta + (\alpha(E_{x_0}) - \beta(E_{x_0}))\eta_{x_0} \in (T_{x_0}S)^0$ satisfies (29). □

Remark 4.5. Actually, theorem 3.15 is a corollary of theorem 4.4.

5. Conformal Jacobi morphisms and co-isotropic submanifolds

In this section, we will study conformal Jacobi morphisms and we will obtain a generalization of theorems 2.2 and 3.7.

For this purpose, we prove the following.

Proposition 5.1. If (M_1, Λ_1, E_1) and (M_2, Λ_2, E_2) are Jacobi manifolds, t is the usual coordinate on $\mathbb{R}$ and Λ and E are the 2-vector and the vector field, respectively, on the product manifold $M_1 \times M_2 \times \mathbb{R}$ given by

$$\Lambda = \Lambda_1 + \frac{\partial}{\partial t} \wedge E_1 - \mathrm{e}^t \left(\Lambda_2 - \frac{\partial}{\partial t} \wedge E_2\right) \qquad E = E_1 \tag{30}$$

then $(M_1 \times M_2 \times \mathbb{R}, \Lambda, E)$ is a Jacobi manifold.

Proof. It follows from a direct computation using (6). □

Remark 5.2. If M_1 and M_2 are Poisson manifolds then, from proposition 5.1, we deduce that the 2-vector Λ given by $\Lambda = \Lambda_1 - \mathrm{e}^t \Lambda_2$ defines a Poisson structure on the product manifold $M_1 \times M_2 \times \mathbb{R}$. Moreover, the natural restriction Λ_N of Λ to the submanifold $N = M_1 \times M_2 \times \{0\}$ also defines a Poisson structure on N. In fact, the identification

$$M_1 \times M_2 \to M_1 \times M_2 \times \{0\} \qquad (x_1, x_2) \to (x_1, x_2, 0)$$

is a Poisson isomorphism between the Poisson manifolds $(M_1 \times M_2, \bar{\Lambda})$ and (N, Λ_N), where $\bar{\Lambda} = \Lambda_1 - \Lambda_2$.

Remarks 3.6 and 5.2 and the following result (announced at the beginning of this section) justify the definition of the Jacobi structure (Λ, E) given by (30).

Theorem 5.3. Let $\phi : (M_1, \Lambda_1, E_1) \to (M_2, \Lambda_2, E_2)$ be a differentiable mapping between the Jacobi manifolds (M_1, Λ_1, E_1) and (M_2, Λ_2, E_2). Suppose that a is a positive function on M_1 and that S is the submanifold of the product manifold $M_1 \times M_2 \times \mathbb{R}$ defined by

$$S = \left\{ \left(x_1, \phi(x_1), \ln \frac{1}{a(x_1)} \right) \in M_1 \times M_2 \times \mathbb{R} / x_1 \in M_1 \right\}. \tag{31}$$

Then, the pair (a, ϕ) is a conformal Jacobi morphism if and only if S is a co-isotropic submanifold of $(M_1 \times M_2 \times \mathbb{R}, \Lambda, E)$, where (Λ, E) is the Jacobi structure on $M_1 \times M_2 \times \mathbb{R}$ given by (30).

Proof. We consider the mapping $\tilde{\phi}_a : M_1 \times \mathbb{R} \to M_2 \times \mathbb{R}$ defined by

$$\tilde{\phi}_a(x_1, p) = (\phi(x_1), p + \ln(a(x_1))) \tag{32}$$

for $x_1 \in M_1$ and $p \in \mathbb{R}$.

From (2), (9), (27) and (32), we deduce that the pair (a, ϕ) is a conformal Jacobi morphism if and only if the mapping $\tilde{\phi}_a$ is a Poisson morphism between the Poissonizations $(M_1 \times \mathbb{R}, \tilde{\Lambda}_1)$ and $(M_2 \times \mathbb{R}, \tilde{\Lambda}_2)$ of the Jacobi manifolds M_1 and M_2, respectively (see theorem 3.16).

Now, denote by $\psi : (M_1 \times M_2 \times \mathbb{R}) \times \mathbb{R} \to (M_1 \times \mathbb{R}) \times (M_2 \times \mathbb{R})$ the diffeomorphism given by

$$\psi((x_1, x_2, t), s) = ((x_1, s), (x_2, s - t)) \tag{33}$$

for $x_1 \in M_1$, $x_2 \in M_2$ and $t, s \in \mathbb{R}$.

Let $((M_1 \times M_2 \times \mathbb{R}) \times \mathbb{R}, \tilde{\Lambda})$ be the Poissonization of the Jacobi manifold $(M_1 \times M_2 \times \mathbb{R}, \Lambda, E)$. Then, using (27), (30) and (33), we obtain that ψ is a Poisson isomorphism between the Poisson manifolds $((M_1 \times M_2 \times \mathbb{R}) \times \mathbb{R}, \tilde{\Lambda})$ and $((M_1 \times \mathbb{R}) \times (M_2 \times \mathbb{R}), \tilde{\Lambda}_1 - \tilde{\Lambda}_2)$. Moreover, from (31)–(33), we have that

$$\psi(S \times \mathbb{R}) = \mathrm{Graph}\ \tilde{\phi}_a.$$

Thus, using theorems 2.2 and 4.4, we prove our result. □

Remark 5.4. Let (M, η) be a $(2m+1)$-dimensional contact manifold and S a submanifold of M. Then, S is a Legendre submanifold of M if and only if S is co-isotropic and $\dim S = m$ (see (13) and proposition 3.2). Therefore, theorem 5.3 generalizes theorem 3.7.

From theorem 5.3, we conclude

Corollary 5.5. Let $\phi : (M_1, \Lambda_1, E_1) \to (M_2, \Lambda_2, E_2)$ be a differentiable mapping between the Jacobi manifolds (M_1, Λ_1, E_1) and (M_2, Λ_2, E_2). Then, ϕ is a Jacobi morphism if and only if Graph $\phi \times \{0\}$ is a co-isotropic submanifold of $(M_1 \times M_2 \times \mathbb{R}, \Lambda, E)$, where (Λ, E) is the Jacobi structure on $M_1 \times M_2 \times \mathbb{R}$ given by (30).

Remark 5.6. Using corollary 5.5 and remarks 4.2 and 5.2, we directly deduce theorem 2.2.

6. Conformal Jacobi infinitesimal transformations and Legendre–Lagrangian submanifolds

In this section, we will study conformal Jacobi infinitesimal transformations and we will obtain a generalization of theorems 2.3 and 3.13.

For this purpose, we prove:

Proposition 6.1. Let (M, Λ, E) be a Jacobi manifold and $\bar{\Lambda}$ the 2-vector on the product manifold $\mathbb{R} \times TM$ defined by

$$\bar{\Lambda} = \Lambda^{c} + \frac{\partial}{\partial s} \wedge E^{c} - s\left(\Lambda^{v} + \frac{\partial}{\partial s} \wedge E^{v}\right) \tag{34}$$

where s is the usual coordinate on $\mathbb{R}$, Λ^{c} (respectively, Λ^{v}) is the complete (respectively, vertical) lift to TM of Λ and E^{c} (respectively, E^{v}) is the complete (respectively, vertical) lift to TM of E. Then, the pair $(\bar{\Lambda}, E^{v})$ is a Jacobi structure on $\mathbb{R} \times TM$.

Proof. Using (6) and the results of [17], we deduce that

$$\begin{aligned} &[\Lambda^{c}, \Lambda^{c}] = [\Lambda, \Lambda]^{c} = 2(E^{c} \wedge \Lambda^{v} + E^{v} \wedge \Lambda^{c}) \qquad [\Lambda^{v}, \Lambda^{v}] = 0 \\ &[\Lambda^{c}, \Lambda^{v}] = [\Lambda, \Lambda]^{v} = 2E^{v} \wedge \Lambda^{v}. \end{aligned} \tag{35}$$

We also have

$$\begin{aligned} &[\Lambda^{c}, E^{c}] = [\Lambda, E]^{c} = 0 \qquad [\Lambda^{c}, E^{v}] = [\Lambda, E]^{v} = 0 \\ &[\Lambda^{v}, E^{c}] = [\Lambda, E]^{v} = 0 \qquad [\Lambda^{v}, E^{v}] = 0. \end{aligned} \tag{36}$$

Thus, from (34)–(36), we conclude that the pair $(\bar{\Lambda}, E^{v})$ is a Jacobi structure on $\mathbb{R} \times TM$. □

Remark 6.2. If (M, Λ) is a Poisson manifold then, using proposition 6.1, we obtain that the 2-vector $\bar{\Lambda} = \Lambda^{c} - s\Lambda^{v}$ defines a Poisson structure on $\mathbb{R} \times TM$. Moreover, the natural restriction $\bar{\Lambda}_N$ of $\bar{\Lambda}$ to the submanifold $N = \{0\} \times TM$ also defines a Poisson structure on N. In fact, the diffeomorphism

$$TM \to \{0\} \times TM \qquad v \to (0, v)$$

is a Poisson isomorphism between the Poisson manifolds (TM, Λ^{c}) and $(N, \bar{\Lambda}_N)$.

If $f \in C^{\infty}(M, \mathbb{R})$, we will denote by f^{c} (respectively, f^{v}) the complete (respectively, vertical) lift to TM of f. Then, from (7), (34) and the results of [17], we have:

Corollary 6.3. Let (M, Λ, E) be a Jacobi manifold, $\bar{\Lambda}$ the 2-vector on $\mathbb{R} \times TM$ given by (34) and E^{v} the vertical lift to TM of E. Suppose that $\{\, ,\}_{\mathbb{R}\times TM}$ (respectively, $\{\, ,\}_M$) is the Jacobi bracket of $(\mathbb{R} \times TM, \bar{\Lambda}, E^{\mathrm{v}})$ (respectively, (M, Λ, E)). Then

$$\{f^{\mathrm{c}}, g^{\mathrm{c}}\}_{\mathbb{R}\times TM} - \{f, g\}^{\mathrm{c}}_M + s\{f, g\}^{\mathrm{v}}_M = f^{\mathrm{v}}(E(g))^{\mathrm{c}} - g^{\mathrm{v}}(E(f))^{\mathrm{c}} + sg^{\mathrm{v}}(E(f))^{\mathrm{v}} - sf^{\mathrm{v}}(E(g))^{\mathrm{v}}$$
$$\{f^{\mathrm{c}}, g^{\mathrm{v}}\}_{\mathbb{R}\times TM} - (1-s)\{f, g\}^{\mathrm{v}}_M = (s-1)f^{\mathrm{v}}(E(g))^{\mathrm{v}} - sg^{\mathrm{v}}(E(f))^{\mathrm{v}}$$
$$\{f^{\mathrm{v}}, g^{\mathrm{v}}\}_{\mathbb{R}\times TM} = 0 \qquad \{s, f^{\mathrm{c}}\}_{\mathbb{R}\times TM} = (E(f))^{\mathrm{c}} \qquad \{s, f^{\mathrm{v}}\}_{\mathbb{R}\times TM} = (E(f))^{\mathrm{v}}$$

for $f, g \in C^\infty(M, \mathbb{R})$.

Using corollary 6.3, we deduce:

Corollary 6.4. Let (M, Λ, E) be a Jacobi manifold, $\bar{\Lambda}$ the 2-vector on $\mathbb{R} \times TM$ given by (34) and E^{v} the vertical lift to TM of E. If $\{\, ,\}_{\mathbb{R}\times TM}$ (respectively, $\{\, ,\}_M$) is the Jacobi bracket of $(\mathbb{R} \times TM, \bar{\Lambda}, E^{\mathrm{v}})$ (respectively, (M, Λ, E)) and $f, g \in C^\infty(M, \mathbb{R})$ are basic functions (that is, $E(f) = E(g) = 0$), then

$$\{f^{\mathrm{c}}, g^{\mathrm{c}}\}_{\mathbb{R}\times TM} = \{f, g\}^{\mathrm{c}}_M - s\{f, g\}^{\mathrm{v}}_M$$
$$\{f^{\mathrm{c}}, g^{\mathrm{v}}\}_{\mathbb{R}\times TM} = (1-s)\{f, g\}^{\mathrm{v}}_M$$
$$\{f^{\mathrm{v}}, g^{\mathrm{v}}\}_{\mathbb{R}\times TM} = \{s, f^{\mathrm{c}}\}_{\mathbb{R}\times TM} = \{s, f^{\mathrm{v}}\}_{\mathbb{R}\times TM} = 0.$$

Remarks 3.12 and 6.2 and the following result (announced at the beginning of this section) justify the definition of the Jacobi structure $(\bar{\Lambda}, E^{\mathrm{v}})$ on the product manifold $\mathbb{R} \times TM$.

Theorem 6.5. Let (M, Λ, E) be a Jacobi manifold. Suppose that X is a vector field on M and denote by $f \times X : M \to \mathbb{R} \times TM$ the mapping

$$x \in M \to (f \times X)(x) = (f(x), X(x)) \in \mathbb{R} \times TM \tag{37}$$

f being a C^∞ function on M. Then, the pair (f, X) is a conformal Jacobi infinitesimal transformation if and only if $(f \times X)(M)$ is a Legendre–Lagrangian submanifold of the Jacobi manifold $(\mathbb{R} \times TM, \bar{\Lambda}, E^{\mathrm{v}})$, where E^{v} is the vertical lift to TM of E and $\bar{\Lambda}$ is the 2-vector on $\mathbb{R} \times TM$ given by (34).

Proof. Let $(M \times \mathbb{R}, \tilde{\Lambda})$ be the Poissonization of M.

Using the results of [9] we have that the pair (f, X) is a conformal Jacobi infinitesimal transformation if and only if the vector field $\tilde{X}_f = X + f\partial/\partial t$ is a Poisson infinitesimal transformation of the Poisson manifold $(M \times \mathbb{R}, \tilde{\Lambda})$ (see theorem 3.17).

Now, denote by $\psi : (\mathbb{R} \times TM) \times \mathbb{R} \to T(M \times \mathbb{R})$ the diffeomorphism defined by

$$\psi((s, v), p) = v + s\left.\frac{\partial}{\partial t}\right|_p \tag{38}$$

for $v \in TM$ and $s, p \in \mathbb{R}$.

If $((\mathbb{R}\times TM)\times\mathbb{R}, \tilde{\bar{\Lambda}})$ is the Poissonization of the Jacobi manifold $(\mathbb{R}\times TM, \bar{\Lambda}, E^{\mathrm{v}})$ then, using (27), (34), (38) and the results of [17], we deduce that ψ is a Poisson isomorphism between the Poisson manifolds $((\mathbb{R} \times TM) \times \mathbb{R}, \tilde{\bar{\Lambda}})$ and $(T(M \times \mathbb{R}), \tilde{\Lambda}^{\mathrm{c}})$. Furthermore, from (37) and (38), we obtain that

$$\psi((f \times X)(M) \times \mathbb{R}) = \tilde{X}_f(M \times \mathbb{R}).$$

Therefore, using theorems 2.3 and 4.4, we prove our result. □

Remark 6.6. Theorem 6.5 generalizes theorem 3.13.

From theorem 6.5, we conclude:

Corollary 6.7. Let (M, Λ, E) be a Jacobi manifold and X a vector field on M. Then, X is a Jacobi infinitesimal transformation if and only if $\{0\} \times X(M)$ is a Legendre–Lagrangian submanifold of the Jacobi manifold $(\mathbb{R} \times TM, \bar{\Lambda}, E^{\mathrm{v}})$, where E^{v} is the vertical lift to TM of E and $\bar{\Lambda}$ is the 2-vector on $\mathbb{R} \times TM$ given by (34).

Remark 6.8. Using corollary 6.7 and remarks 4.2 and 6.2, we directly deduce the second part of theorem 2.3.

From (10)–(12), theorem 6.5 and corollary 6.7, we obtain:

Corollary 6.9. Let (M, Λ, E) be a Jacobi manifold and f a C^∞-function on M. Suppose that E^{v} is the vertical lift to TM of E and that $\bar{\Lambda}$ is the 2-vector on $\mathbb{R} \times TM$ given by (34).

(i) If X_f is the Hamiltonian vector field associated with f then $(-E(f) \times X_f)(M)$ is a Legendre–Lagrangian submanifold of the Jacobi manifold $(\mathbb{R} \times TM, \bar{\Lambda}, E^{\mathrm{v}})$.

(ii) If f is a basic function then $\{0\} \times X_f(M)$ is a Legendre–Lagrangian submanifold of the Jacobi manifold $(\mathbb{R} \times TM, \bar{\Lambda}, E^{\mathrm{v}})$.

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